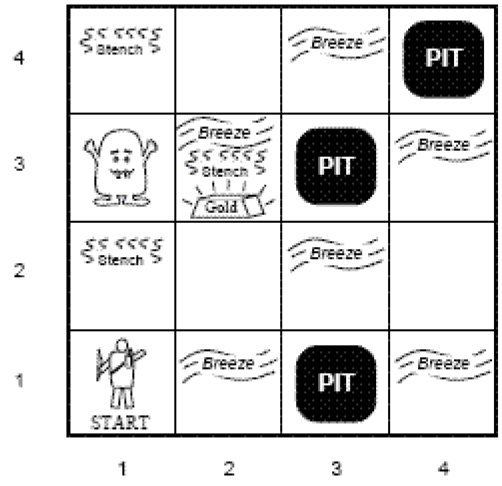
INFERENCE IN PROPOSITIONAL LOGIC



Outline

- Inference rules and theorem proving
  - Forward chaining
  - Backward chaining
  - Resolution

Proof Methods

- Proof methods divide into (roughly) two kinds:
- Application of inference rules
  - Legitimate (sound) generation of new sentences from old
  - Proof = a sequence of inference rule applications
    - Can use inference rules as operators in a standard search algorithm
  - Typically require translation of sentences into a normal form
- Model checking
  - Truth table enumeration (always exponential in n)
  - Heuristic search in model space (sound but incomplete)
    - e.g., min-conflicts-like hill-climbing algorithms

Forward and Backward Chaining

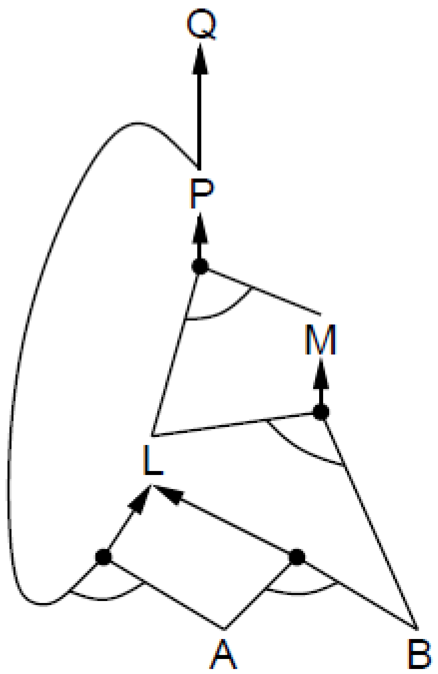
- Horn Form (restricted)
  - KB = conjunction of Horn clauses
  - Horn clause =
    - proposition symbol; or
    - (conjunction of symbols) ⇒ symbol
  - E.g., C ∧ (B ⇒ A) ∧ (C ∧ D) ⇒ B)
- Modus Ponens (for Horn Form): complete for Horn KBs

$$\frac{\alpha_1, \ldots, \alpha_n, \qquad \alpha_1 \wedge \cdots \wedge \alpha_n \Rightarrow \beta}{\beta}$$

- Can be used with forward chaining or backward chaining.
- These algorithms are very natural and run in linear time

Forward Chaining

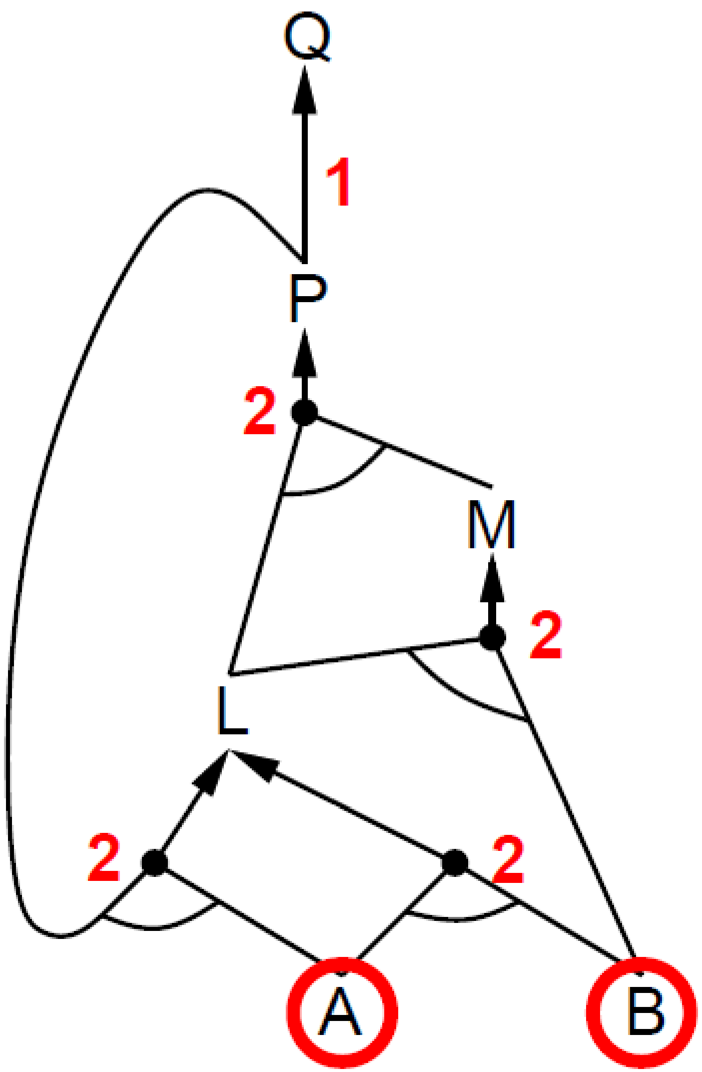
- Idea: fire any rule whose premises are satisfied in the KB, add its conclusion to the KB, until query is found
  - $P \Rightarrow Q$
  - $L \wedge M \Rightarrow P$
  - $B \wedge L \Rightarrow M$
  - $A \wedge P \Rightarrow L$
  - $A \wedge B \Rightarrow L$
  - A
  - B



Forward Chaining Algorithm

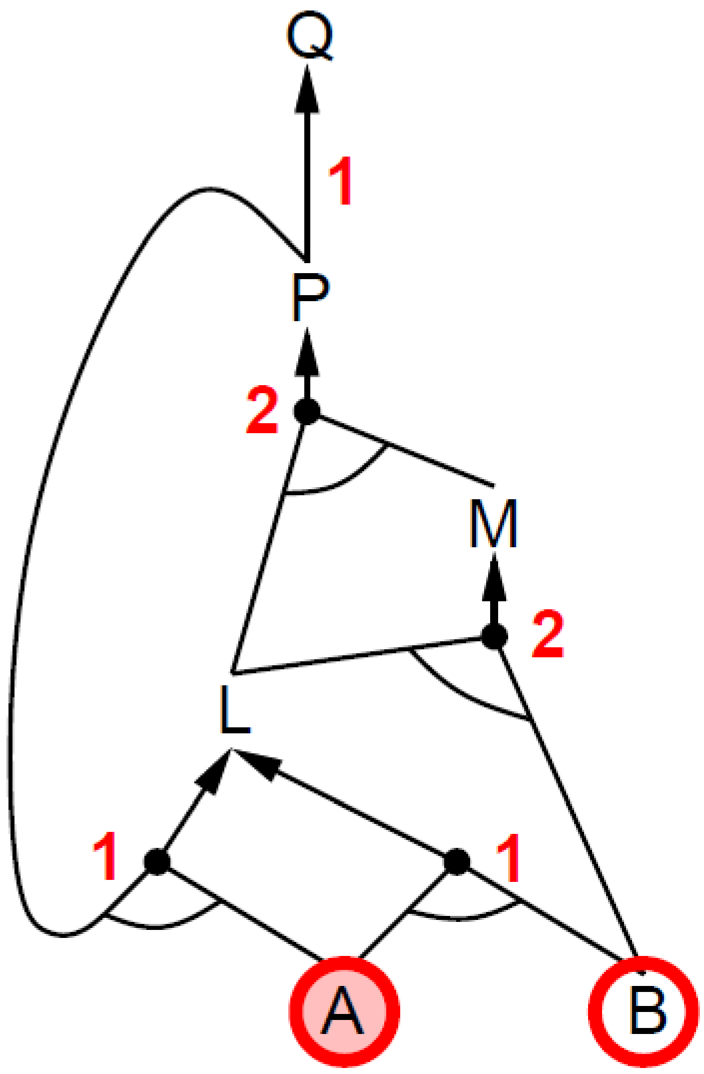
```
function PL-FC-ENTAILS?(KB, q) returns true or false
   inputs: KB, the knowledge base, a set of propositional Horn clauses
           q, the query, a proposition symbol
  local variables: count, a table, indexed by clause, initially the number of premises
                   inferred, a table, indexed by symbol, each entry initially false
                   agenda, a list of symbols, initially the symbols known in KB

   while agenda is not empty do
       p ← POP(agenda)
       unless inferred[p] do
           inferred[p] ← true
           for each Horn clause c in whose premise p appears do
               decrement count[c]
               if count[c] = 0 then do
                   if HEAD[c] = q then return true
                   PUSH(HEAD[c], agenda)
   return false
```

Forward Chaining Example



P ⇒ Q
L ∧ M ⇒ P
B ∧ L ⇒ M
A ∧ P ⇒ L
A ∧ B ⇒ L
A
B

Forward Chaining Example



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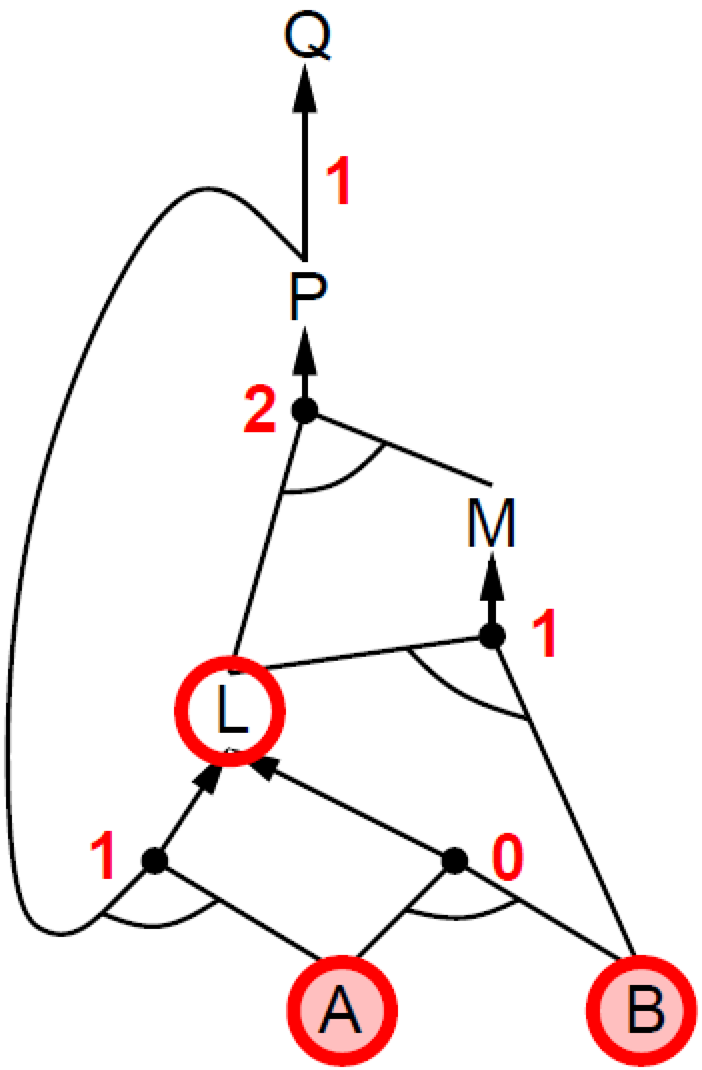

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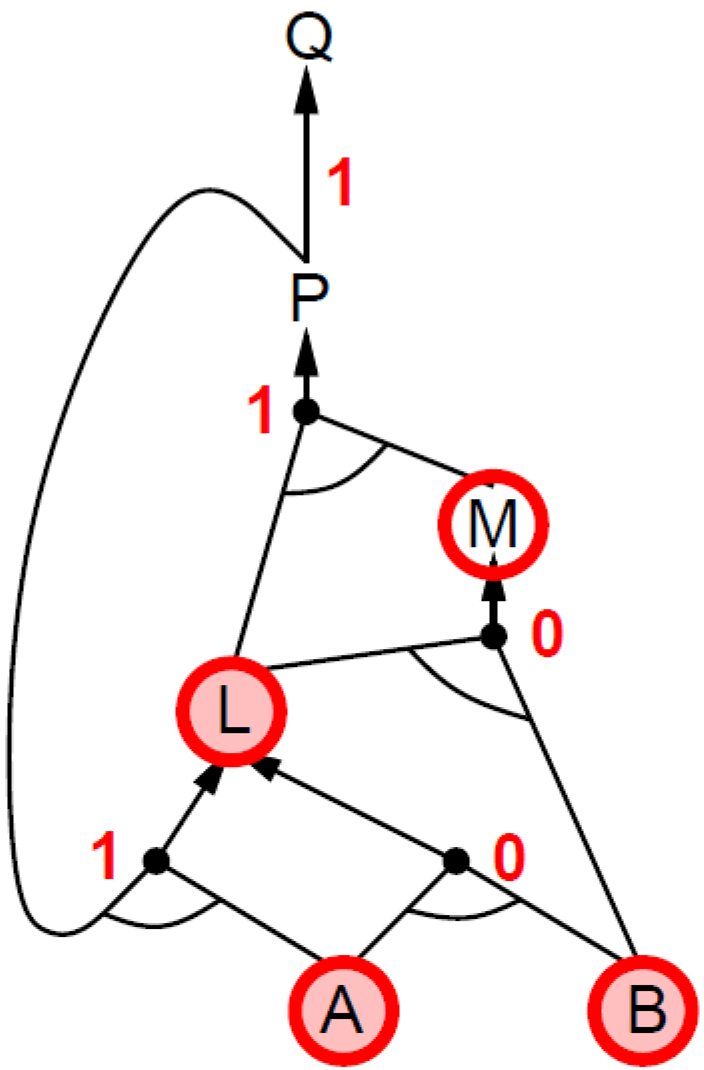

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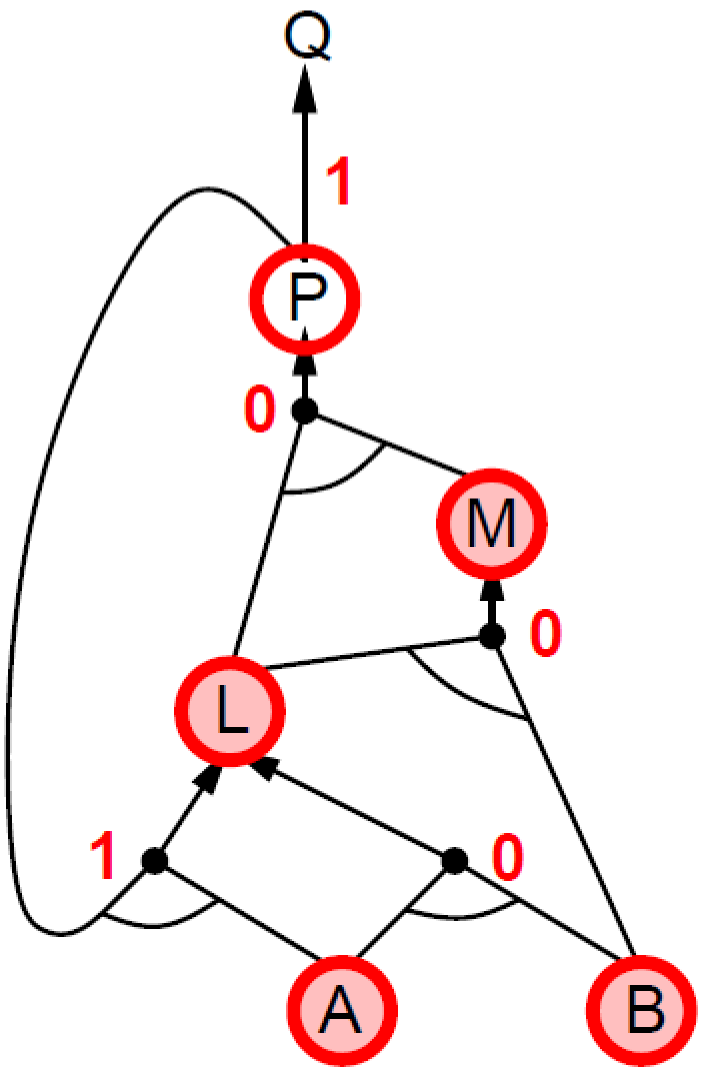

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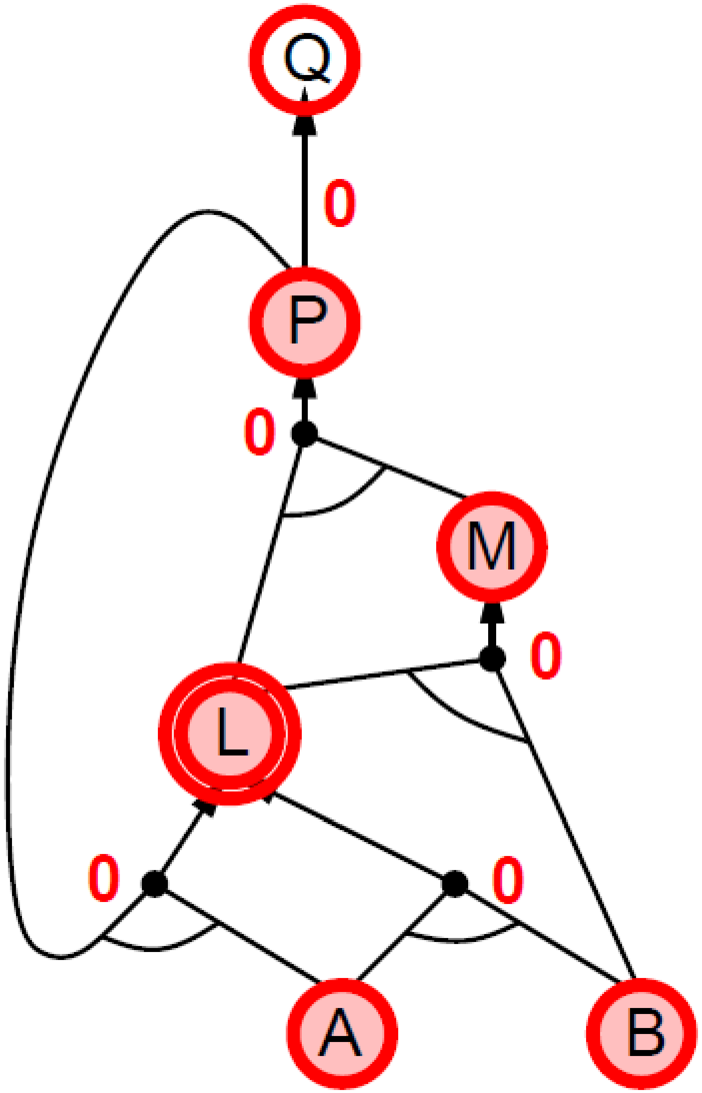

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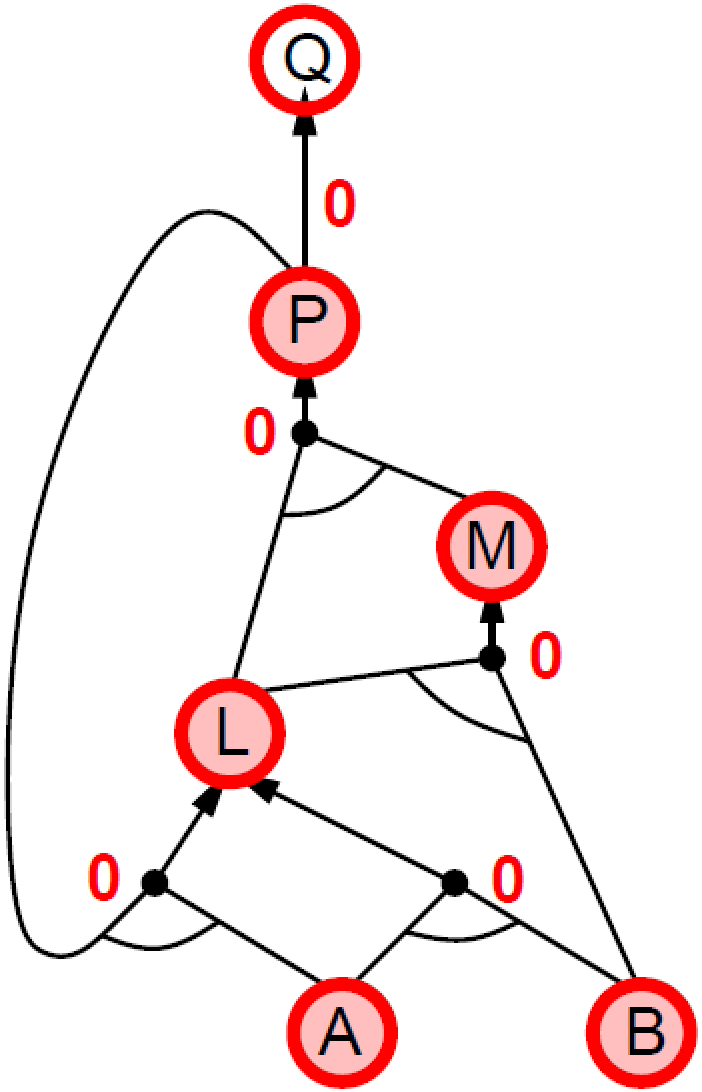

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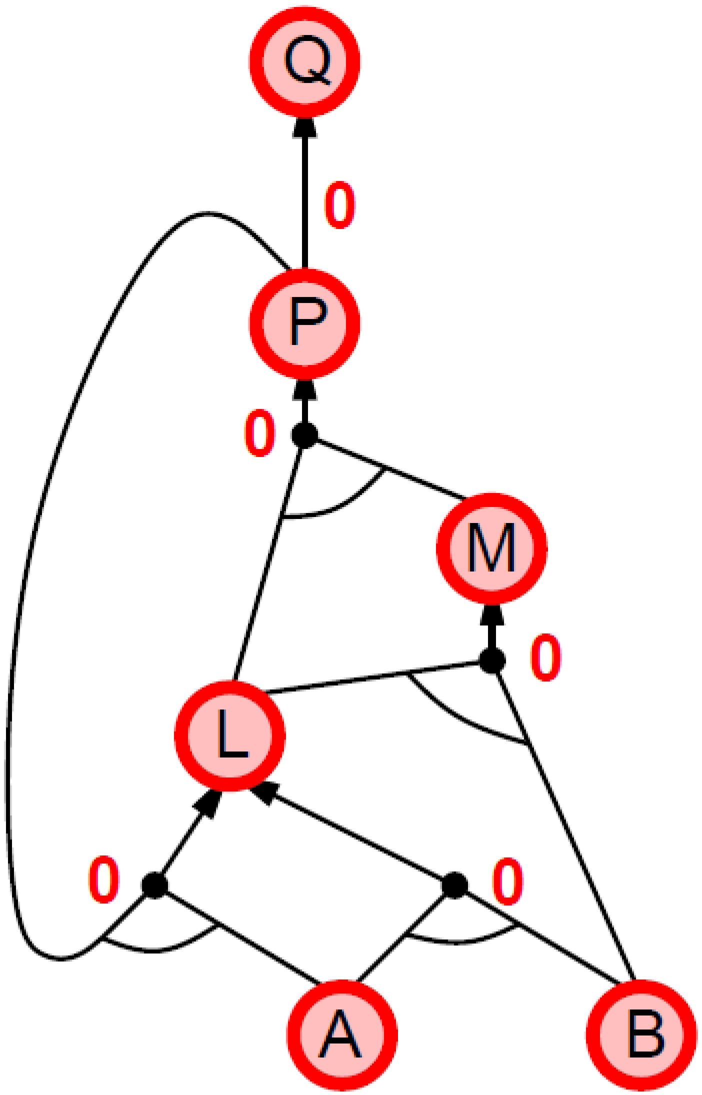

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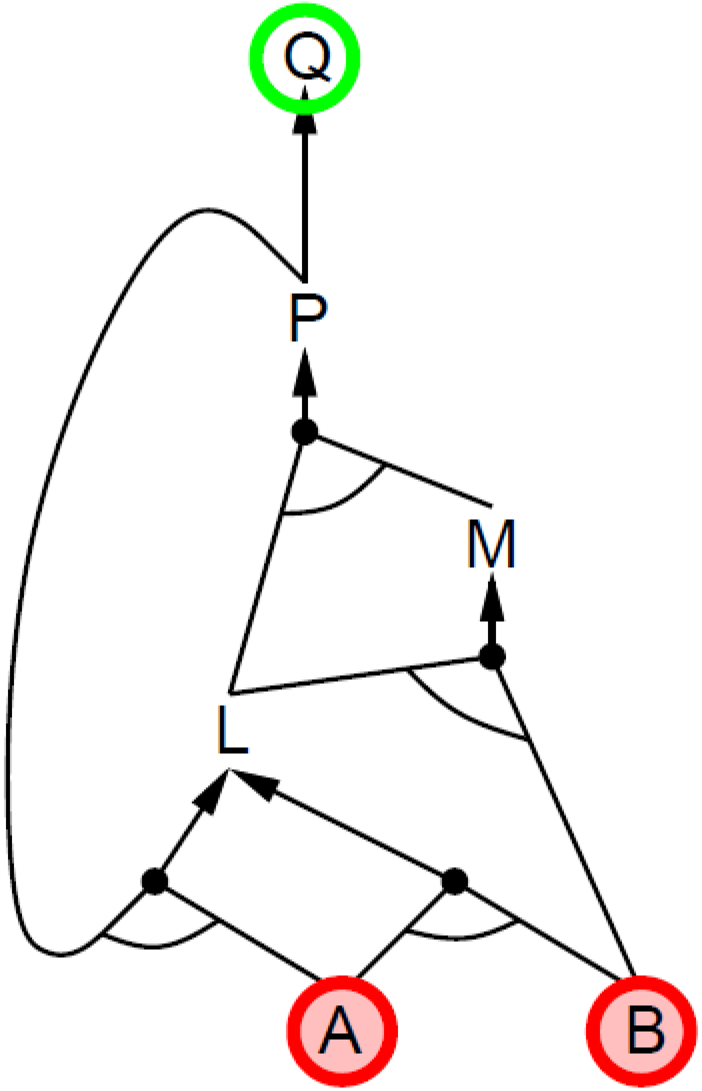
Proof of Completeness

- Forward chaining (FC) derives every atomic sentence that is entailed by KB
- 1. FC reaches a fixed point where no new atomic sentences are derived
- 2. Consider the final state as a model m, assigning true/false to symbols
- 3. Every clause in the original KB is true in m
  - Proof: Suppose a clause $a_1 \land \ldots \land a_k \Rightarrow b$ is false in m
  - Then $a_1 \land \ldots \land a_k$ is true in m and b is false in m
  - Therefore the algorithm has not reached a fixed point!
- 4. Hence m is a model of KB
- 5. If KB $\models$ q, q is true in every model of KB, including m

Backward Chaining

- Idea: work backwards from the query q:
  - To prove q by backward chaining,
    - Check if q is known already, or
    - Prove by backward chaining (BC) all premises of some rule concluding q
- Avoid loops: check if new subgoal is already on the goal stack
- Avoid repeated work: check if new subgoal
  - 1) has already been proved true, or
  - 2) has already failed

Backward Chaining Example



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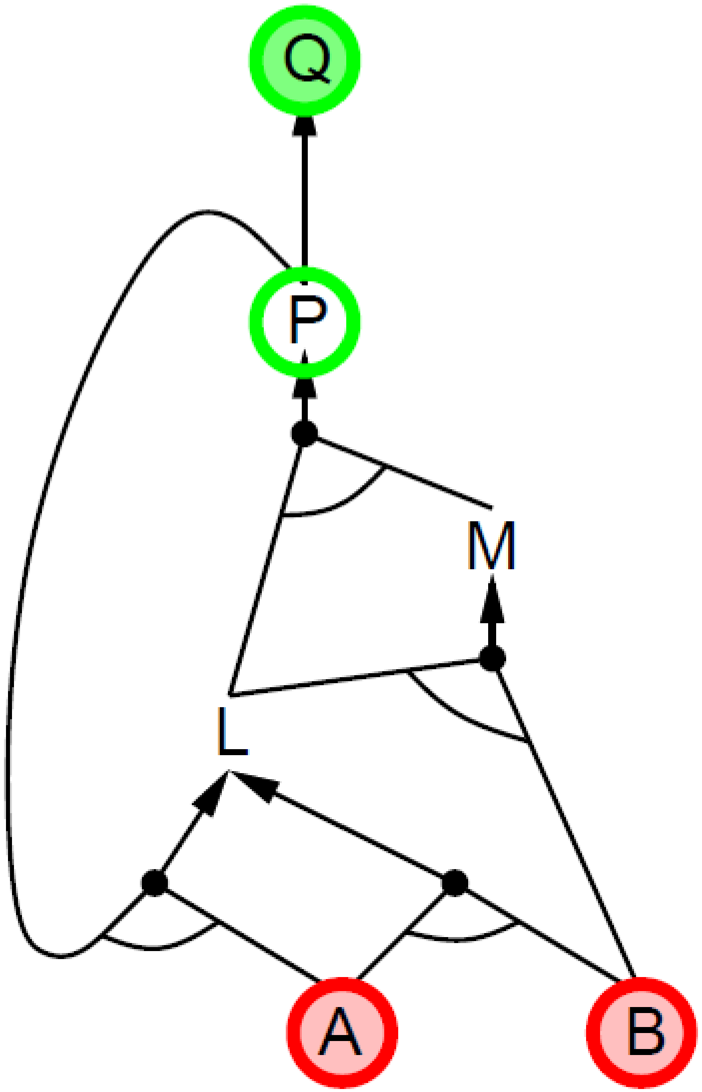

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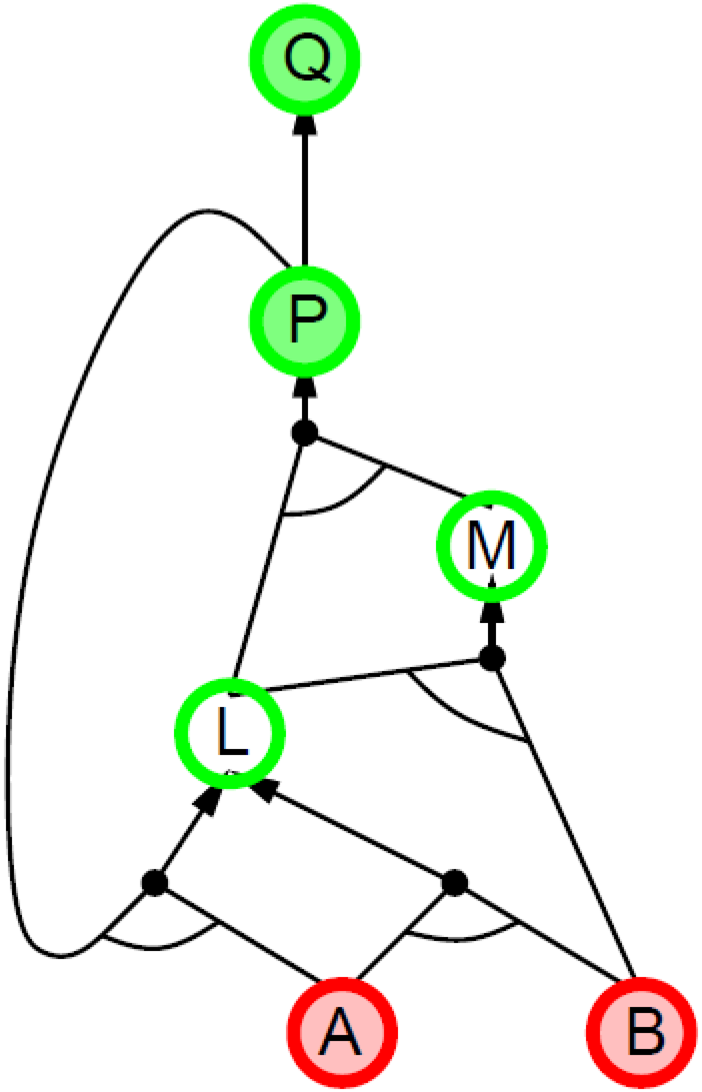

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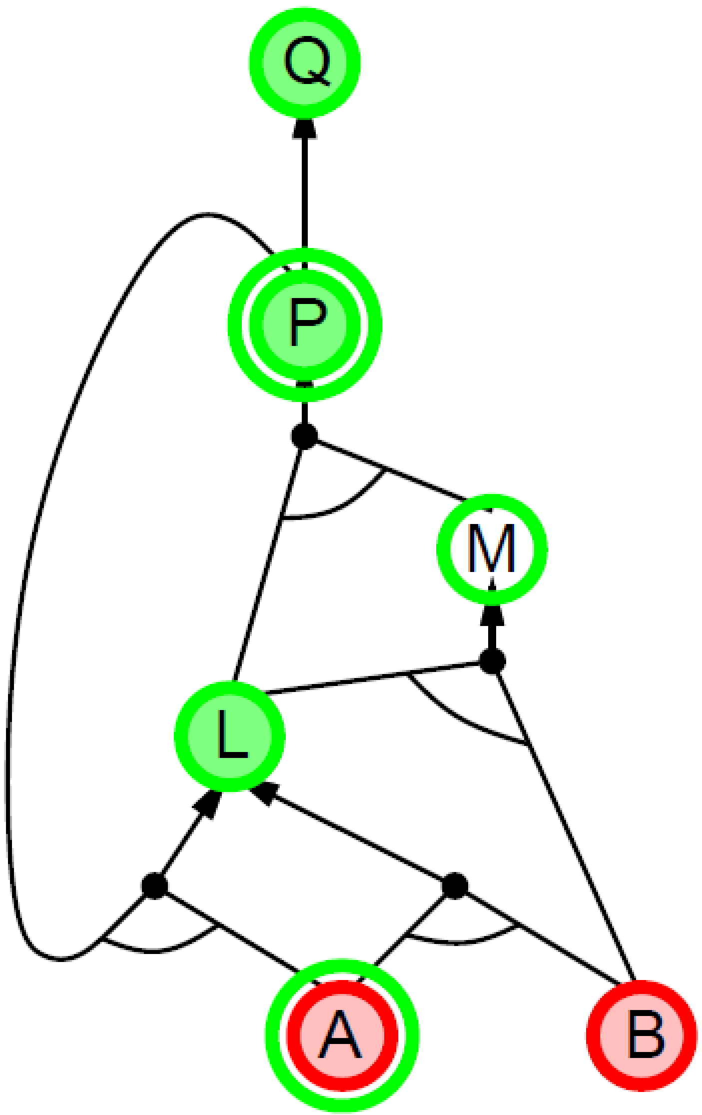

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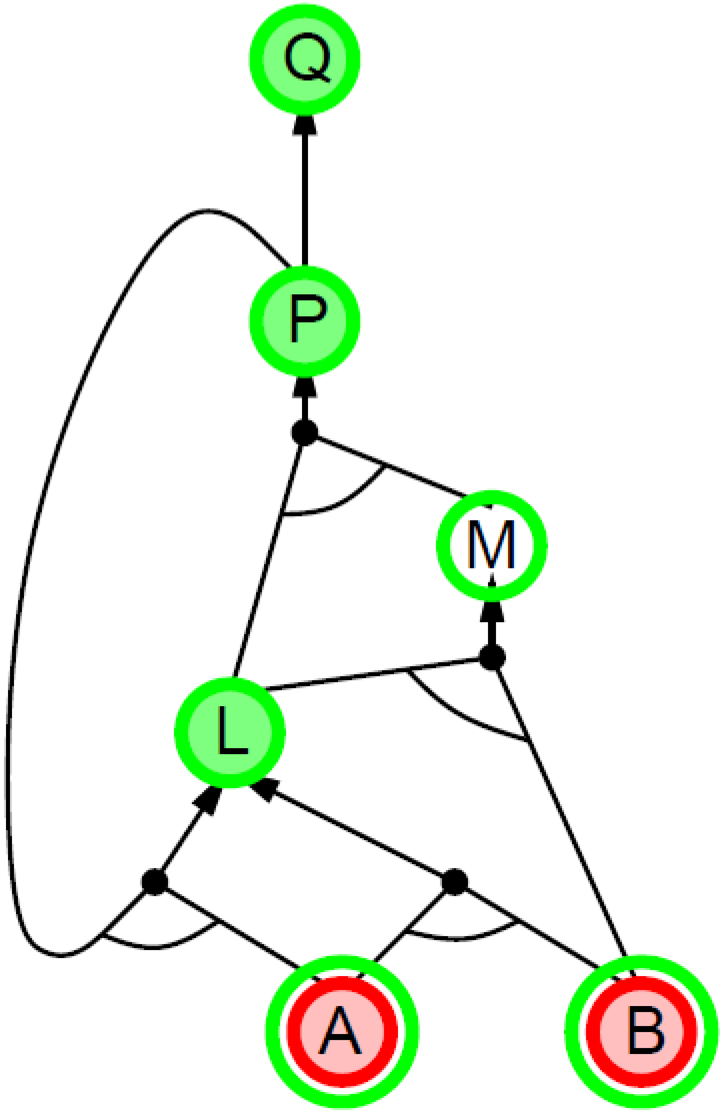

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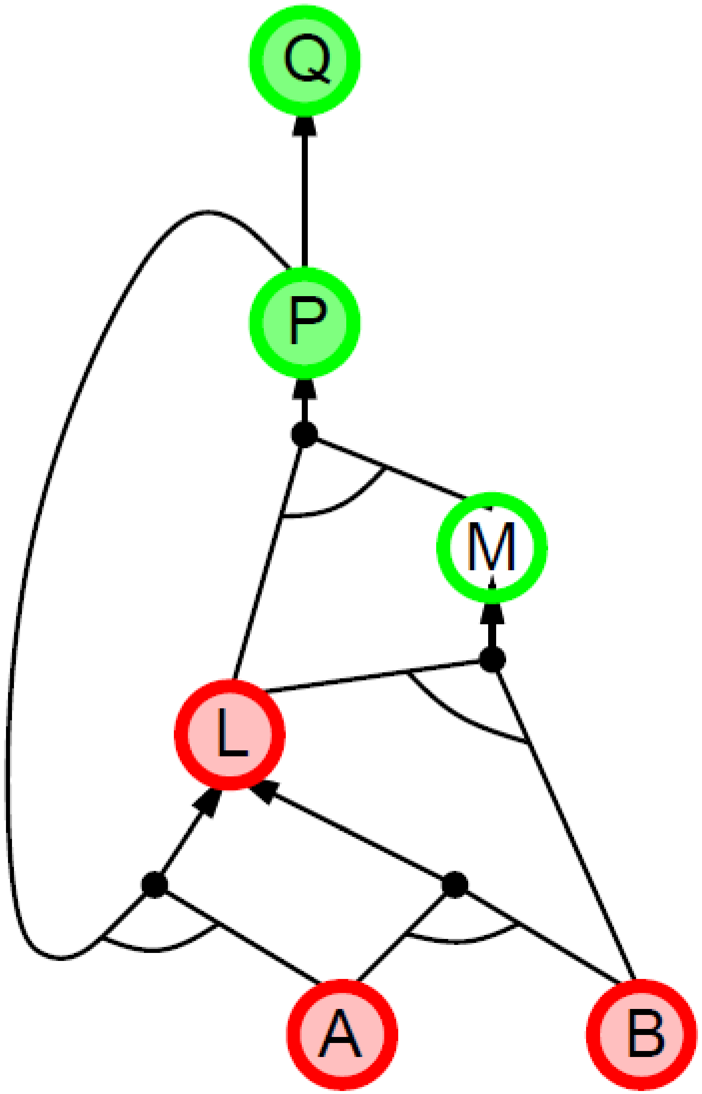

Backward Chaining Example

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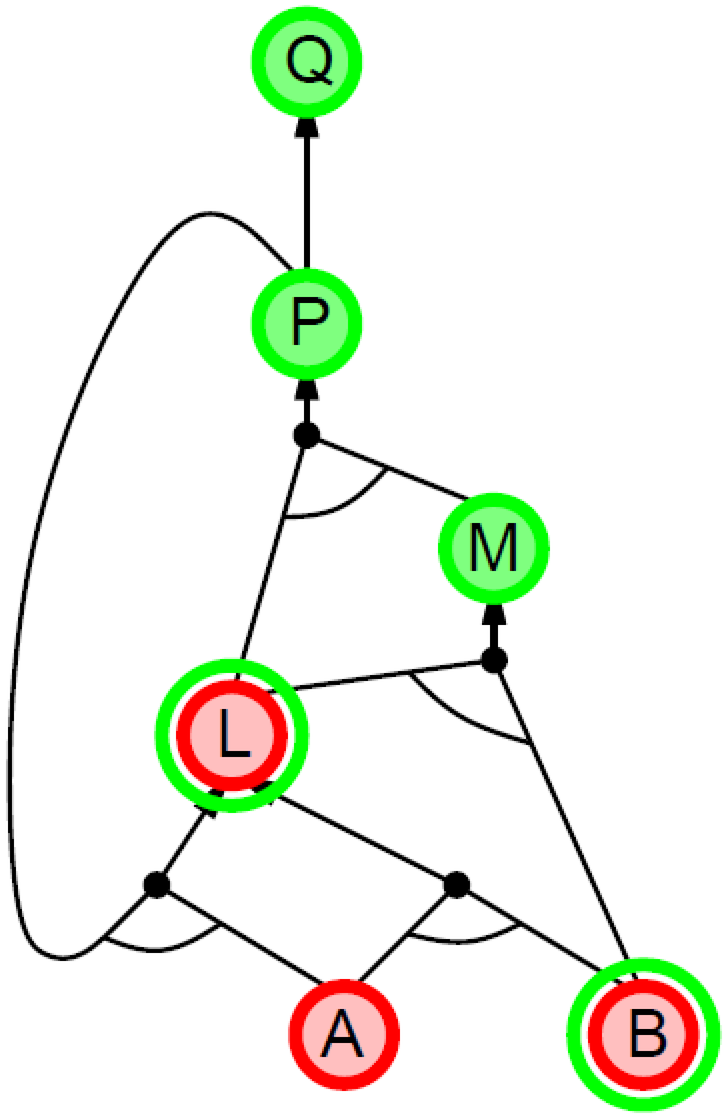

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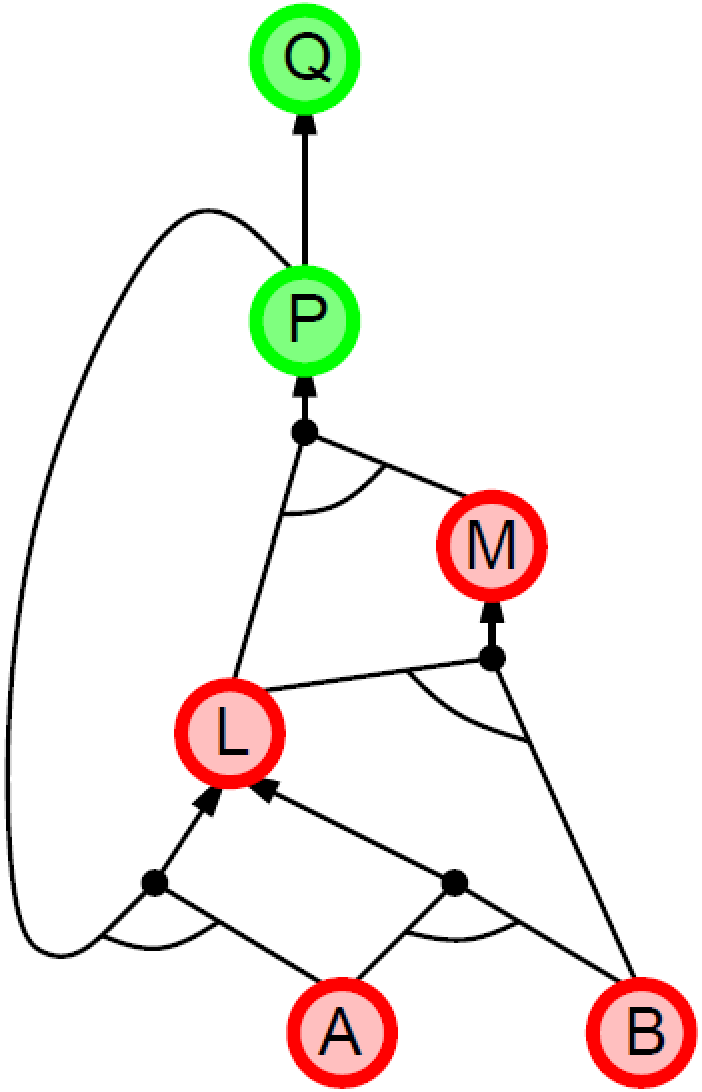

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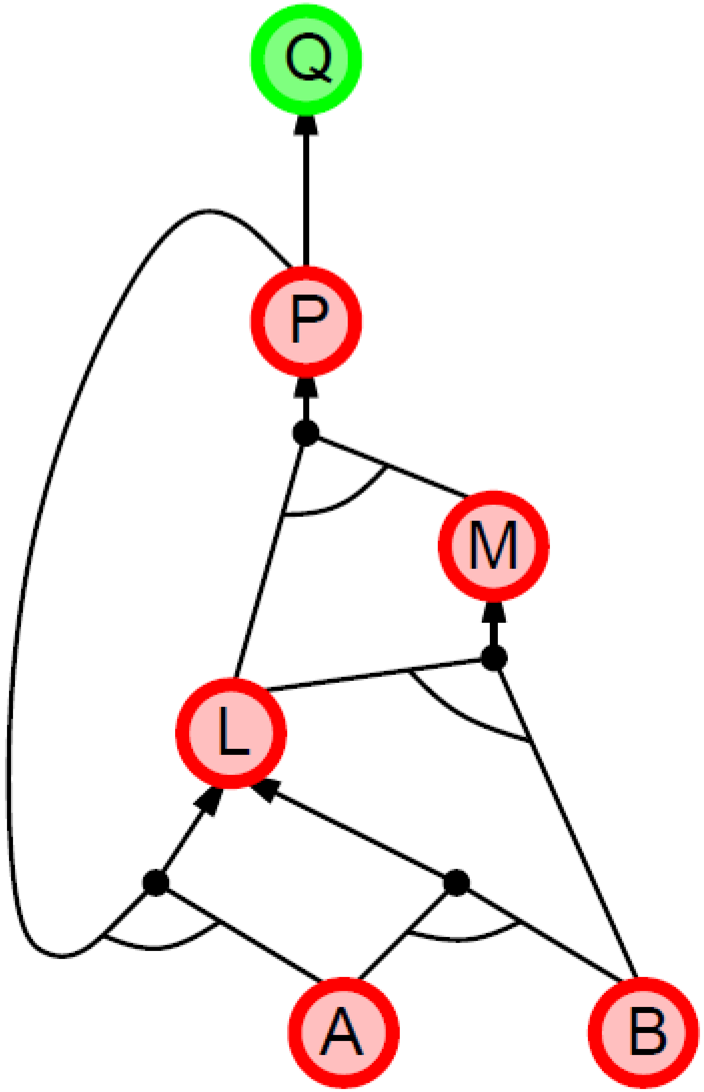

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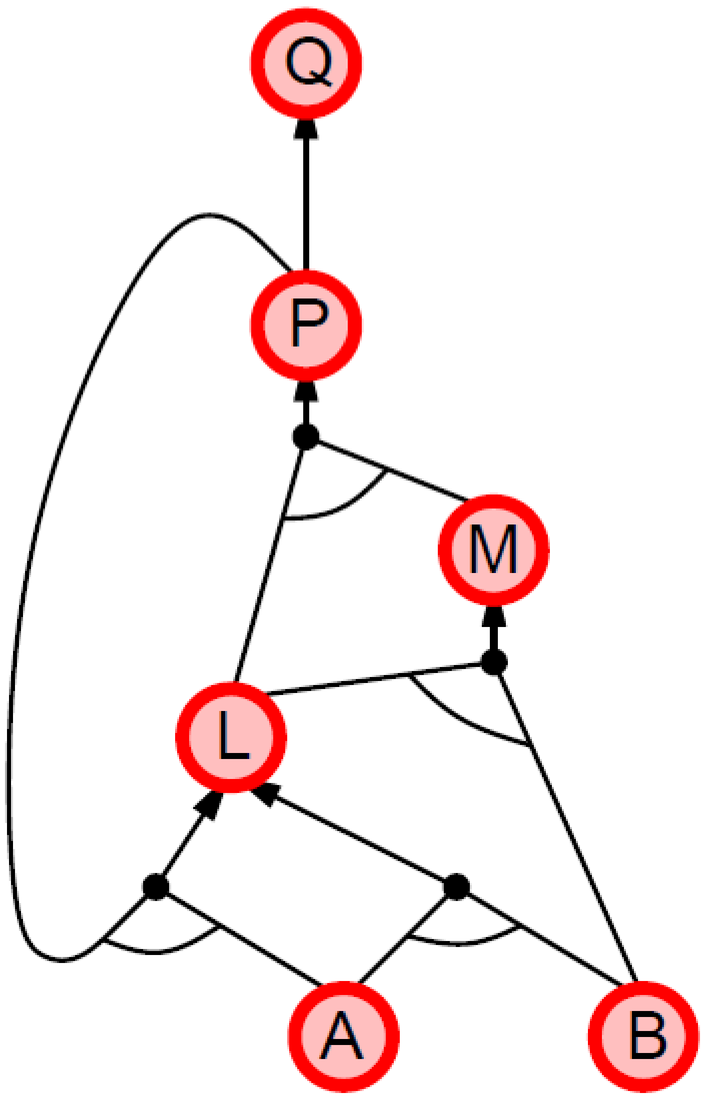

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Forward vs. Backward Chaining

- FC is data-driven, automatic, unconscious processing,
  - e.g., object recognition, routine decisions
- May do lots of work that is irrelevant to the goal
- BC is goal-driven, appropriate for problem-solving,
  - e.g., Where are my keys? How do I get into a PhD program?
- Complexity of BC can be much less than linear in size of KB

Resolution

- Conjunctive Normal Form (CNF - universal)
  - Conjunction of disjunctions of literals
  - Disjunctions of literals means clauses
  - E.g.,

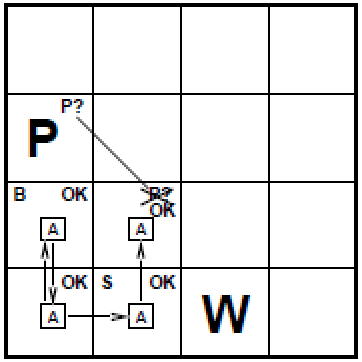
    $(A \lor \neg B) \land (B \lor \neg C \lor \neg D)$

    Resolution inference rule (for CNF): complete for propositional logic

$$\frac{\ell_1 \lor \cdots \lor \ell_k, \qquad m_1 \lor \cdots \lor m_n}{\ell_1 \lor \cdots \lor \ell_{i-1} \lor \ell_{i+1} \lor \cdots \lor \ell_k \lor m_1 \lor \cdots \lor m_{j-1} \lor m_{j+1} \lor \cdots \lor m_n}$$

- where l_i and m_j are complementary literals. E.g.,

$$\frac{P_{1,3} \lor P_{2,2}, \qquad \neg P_{2,2}}{P_{1,3}}$$



- Resolution is sound and complete for propositional logic

Conversion to CNF

$B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})$

1. Eliminate $\Leftrightarrow$, replacing $\alpha \Leftrightarrow \beta$ with $(\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha)$.

   $(B_{1,1} \Rightarrow (P_{1,2} \vee P_{2,1})) \wedge ((P_{1,2} \vee P_{2,1}) \Rightarrow B_{1,1})$

2. Eliminate $\Rightarrow$, replacing $\alpha \Rightarrow \beta$ with $\neg\alpha \vee \beta$.

   $(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg(P_{1,2} \vee P_{2,1}) \vee B_{1,1})$

3. Move $\neg$ inwards using de Morgan's rules and double-negation:

   $(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge ((\neg P_{1,2} \wedge \neg P_{2,1}) \vee B_{1,1})$

4. Apply distributivity law ($\vee$ over $\wedge$) and flatten:

   $(\neg B_{1,1} \vee P_{1,2} \vee P_{2,1}) \wedge (\neg P_{1,2} \vee B_{1,1}) \wedge (\neg P_{2,1} \vee B_{1,1})$

- Proof by contradiction, i.e., show KB ∧ ¬α : unsatisfiable

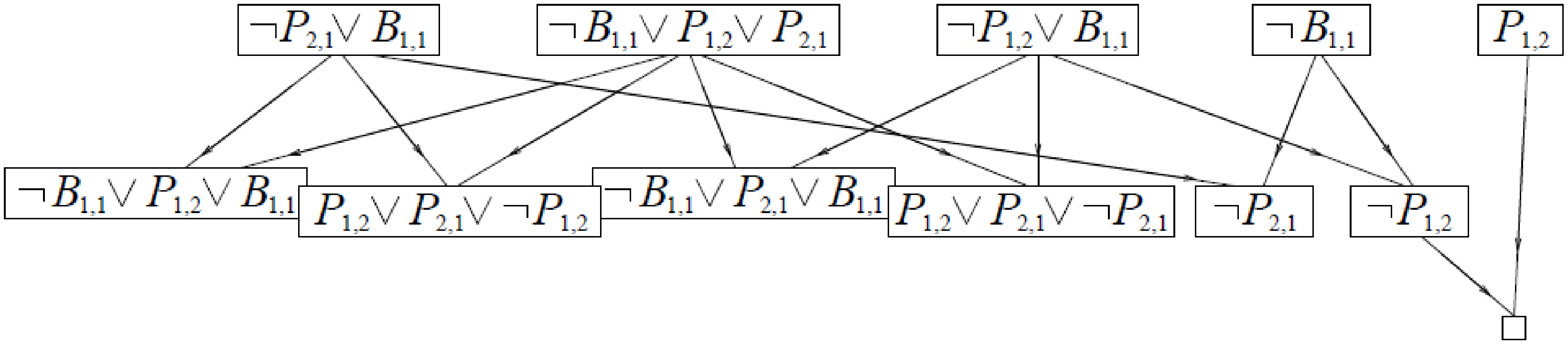
Resolution Algorithm

function PL-RESOLUTION(KB, α) **returns** *true* or *false*
  inputs: KB, the knowledge base, a sentence in propositional logic
    α, the query, a sentence in propositional logic

  $clauses \leftarrow$ the set of clauses in the CNF representation of $KB \wedge \neg\alpha$
  $new \leftarrow \{\}$
  loop do
    for each C_i, C_j **in** *clauses* **do**
      $resolvents \leftarrow$ PL-RESOLVE(C_i, C_j)
      if *resolvents* contains the empty clause **then return** *true*
      $new \leftarrow new \cup resolvents$
    if $new \subseteq clauses$ **then return** *false*
    $clauses \leftarrow clauses \cup new$

Resolution Example

$$KB = (B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})) \wedge \neg B_{1,1} \quad \alpha = \neg P_{1,2}$$



Summary

- Logical agents apply inference to a knowledge base to derive new information and make decisions
- Basic concepts of logic:
  - Syntax: formal structure of sentences
  - Semantics: truth of sentences with respect to models
  - Entailment: necessary truth of one sentence given another
  - Inference: deriving sentences from other sentences
  - Soundness: derivations produce only entailed sentences
  - Completeness: derivations can produce all entailed sentences
- Wumpus world requires the ability to represent partial and negated information, reason by cases, etc.
- Forward, backward chaining are linear-time, complete for Horn clauses
- Resolution is complete for propositional logic
- Propositional logic lacks expressive power

Summary

- Inference rules and theorem proving
  - Forward chaining
  - Backward chaining
  - Resolution